1. Basics

The two-scalar formulation depends on hydrostatic stress p and von Mises equivalent stress q.

$$p=\dfrac{\mathrm{tr}\sigma}{3}=\dfrac{I_1}{3},\qquad q=\sqrt{3J_2}=\sqrt{\dfrac{3}{2}s:s}=\sqrt{\dfrac{3}{2}}|s|. \tag{1}$$

2. Yield Function

The yield function is defined as

$$F=\dfrac{(p-p_t+a)^2}{b^2}+\dfrac{q^2}{M^2}-a^2, \tag{2}$$

where $p_t\geqslant 0$ is the tensile yield strength, $b=1$ if $p_e\geqslant 0$ and $b=\beta$ if $p_e<0$ with $p_e=p-p_t+a$ denotes relative stress to the origin of ellipse, β is a constant that controls the shape of negative-wards half of yielding ellipse, M is the ratio between two radii of yielding ellipse.

3. Flow Rule

The associative plasticity is assumed so that

$$\Delta\varepsilon^p=\Delta\gamma\dfrac{\partial F}{\partial\sigma}=\Delta\gamma\left(\dfrac{2p_e}{3b^2}I+\dfrac{3}{M^2}s\right) \tag{3}$$

via the following relationship

$$\dfrac{\mathrm{d}q^2}{\mathrm{d}\sigma}=\dfrac{3}{2}\dfrac{\mathrm{d}\left(s:s\right)}{\mathrm{d}\sigma}=\dfrac{3}{2}\left(\dfrac{\mathrm{d}s}{\mathrm{d}\sigma}:s+s:\dfrac{\mathrm{d}s}{\mathrm{d}\sigma}\right)=3s.$$

Accordingly,

$$\Delta\varepsilon_v^p=\Delta\gamma\dfrac{2p_e}{b^2},\qquad\Delta\varepsilon_d^p=\Delta\gamma\dfrac{3}{M^2}s, \tag{4}$$

where $\Delta\varepsilon_v^p=\mathrm{tr}\Delta\varepsilon^p$ is the volumetric strain scalar and $\Delta\varepsilon_d^p$ is the deviatoric strain tensor.

4. Hardening Rule

The hardening variable α is defined as the volumetric strain ε_v^p so that

$$\alpha=\varepsilon_v^p. \tag{5}$$

The corresponding incremental form is then

$$\alpha-\alpha_n=\Delta\alpha=\Delta\varepsilon_v^p=\Delta\gamma\dfrac{2p_e}{b^2}, \tag{6}$$

$$\alpha-\alpha_n-\Delta\gamma\dfrac{2p_e}{b^2}=0. \tag{7}$$

The hardening rule is then defined as a function of α,

$$a=a\left(\alpha\right)\geqslant 0. \tag{8}$$

5. Residual

By using the elastic relationship,

$$s=2G\varepsilon_d^e=2G\left(\varepsilon_d^{tr}-\Delta\varepsilon_d^p\right)=s^{tr}-\Delta\gamma\frac{6G}{M^2}s,\tag{9}$$

$$p=K\varepsilon_v^e=K\left(\varepsilon_v^{tr}-\Delta\varepsilon_v^p\right)=p^{tr}-K\left(\alpha-\alpha_n\right).\tag{10}$$

Hence,

$$s=\frac{M^2}{M^2+6G\Delta\gamma}s^{tr},\qquad q=\frac{M^2}{M^2+6G\Delta\gamma}q^{tr}.\tag{11}$$

The governing residual equations for independent variables $x=\begin{bmatrix}\Delta\gamma & \alpha\end{bmatrix}^\mathrm{T}$ can be expressed as

$$R=\left\{\begin{array}{l}\dfrac{p_e^2}{b^2}+\dfrac{q^2}{M^2}-a^2=0,\\[2ex]\alpha-\alpha_n-\Delta\gamma\dfrac{2}{b^2}p_e=0.\end{array}\right.\tag{12}$$

where $p_e=p^{tr}-K\alpha+K\alpha_n-p_t+a$ and $q=\dfrac{M^2}{M^2+6G\Delta\gamma}q^{tr}$.

6. Local Iteration

The Jacobian can be formed accordingly.

$$J=\frac{\mathrm{d}R}{\mathrm{d}x}=\begin{bmatrix}\dfrac{-12GM^2q^{tr,2}}{(M^2+6G\Delta\gamma)^3} & \dfrac{2}{b^2}p_e(a'-K)-2aa'\\ -\dfrac{2}{b^2}p_e & 1-\Delta\gamma\dfrac{2}{b^2}(a'-K)\end{bmatrix}.\tag{13}$$

7. Tangent Stiffness

At local iteration, ε^{tr} is fixed and R is iterated out. Noting that in the global iteration, ε^{tr} is also a variable that changes. If local iteration is converged, then $R=0$, so

$$\frac{\mathrm{d}R}{\mathrm{d}\varepsilon^{tr}}=\frac{\partial R}{\partial\varepsilon^{tr}}+\frac{\partial R}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\varepsilon^{tr}}=0,\tag{14}$$

consequently,

$$\frac{\mathrm{d}x}{\mathrm{d}\varepsilon^{tr}}=\begin{bmatrix}\dfrac{\mathrm{d}\Delta\gamma}{\mathrm{d}\varepsilon^{tr}}\\[2ex]\dfrac{\mathrm{d}\alpha}{\mathrm{d}\varepsilon^{tr}}\end{bmatrix}=-\left(\frac{\partial R}{\partial x}\right)^{-1}\frac{\partial R}{\partial\varepsilon^{tr}}=-J^{-1}\frac{\partial R}{\partial\varepsilon^{tr}}.\tag{15}$$

Taking derivatives about ε^{tr} gives

$$\frac{\partial R}{\partial\varepsilon^{tr}}=\begin{bmatrix}\dfrac{2p_eK}{b^2}I+\dfrac{6G}{M^2+6G\Delta\gamma}s\\[2ex]\dfrac{-2\Delta\gamma K}{b^2}I\end{bmatrix}\tag{16}$$

The stress can be expressed as

$$\sigma = s + pI = \frac{M^2}{M^2 + 6G\Delta\gamma} s^{tr} + \left(p^{tr} - K\left(\alpha - \alpha_n\right)\right) I. \tag{17}$$

Direct differentiation gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = \frac{M^2}{M^2 + 6G\Delta\gamma}\frac{\mathrm{d}s^{tr}}{\mathrm{d}\varepsilon} + s^{tr} \otimes \frac{\mathrm{d}\dfrac{M^2}{M^2 + 6G\Delta\gamma}}{\mathrm{d}\varepsilon} + I \otimes \frac{\mathrm{d}\left(p^{tr} - K\left(\alpha - \alpha_n\right)\right)}{\mathrm{d}\varepsilon} \tag{18}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = \frac{2GM^2}{M^2 + 6G\Delta\gamma} I_d + KI \otimes I - KI \otimes \frac{\mathrm{d}\alpha}{\mathrm{d}\varepsilon} - \frac{6GM^2}{\left(M^2 + 6G\Delta\gamma\right)^2} s^{tr} \otimes \frac{\mathrm{d}\Delta\gamma}{\mathrm{d}\varepsilon}, \tag{19}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = D^e - \frac{12G^2\Delta\gamma}{M^2 + 6G\Delta\gamma} I_d - KI \otimes \frac{\mathrm{d}\alpha}{\mathrm{d}\varepsilon} - \frac{6G}{M^2 + 6G\Delta\gamma} s \otimes \frac{\mathrm{d}\Delta\gamma}{\mathrm{d}\varepsilon}. \tag{20}$$