This document introduces the implementation of the Armstrong–Frederick Steel Model.

1 Yield Function

A von Mises yielding function is used.

$$F = \sqrt{\frac{3}{2}}\left|\eta\right| - k, \tag{1}$$

in which $\eta = s - \beta$ is the shifted stress, s is the stress deviator, β is the back stress and k is the isotropic hardening stress.

2 Flow Rule

The associated plasticity flow is adopted. The plastic strain rate is then

$$\mathrm{d}\varepsilon^p = \gamma\frac{\partial F}{\partial\sigma} = \sqrt{\frac{3}{2}}\gamma n, \tag{2}$$

where $n = \dfrac{\eta}{\left|\eta\right|}$. The corresponding accumulated plastic strain rate is

$$\mathrm{d}p = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon^p : \mathrm{d}\varepsilon^p} = \gamma. \tag{3}$$

3 Hardening Rules

An exponential function with a linear component is used for isotropic hardening stress.

$$k = \sigma_y + k_l p + k_s - k_s e^{-mp}. \tag{4}$$

The corresponding derivative is

$$\frac{\mathrm{d}k}{\mathrm{d}\gamma} = k_l + k_s m e^{-mp}. \tag{5}$$

The rate form of back stress $\beta = \sum\beta^i$ is defined as

$$\mathrm{d}\beta^i = \sqrt{\frac{2}{3}}a^i\mathrm{d}\varepsilon^p - b^i\beta^i\mathrm{d}p.$$

In terms of γ, it is $\mathrm{d}\beta^i = a^i\gamma n - b^i\gamma\beta^i$. The incremental form is thus

$$\beta^i = \beta^i_n + a^i\gamma n - b^i\gamma\beta^i, \qquad \beta^i = \frac{\beta^i_n + a^i\gamma n}{1 + b^i\gamma}. \tag{6}$$

4 Incremental Form

The shifted stress can be computed as

$$\eta = s - \beta = 2GI_d\left(\varepsilon^{tr} - \varepsilon^p_n - \sqrt{\frac{3}{2}}\gamma n\right) - \beta = s^{tr} - \sqrt{6}G\gamma n - \sum\frac{\beta^i_n + a^i\gamma n}{1 + b^i\gamma} \tag{7}$$

with $s^{tr} = 2GI_d\left(\varepsilon^{tr} - \varepsilon^p_n\right)$. Hence,

$$\left|\eta\right|n + \sqrt{6}G\gamma n + \sum\frac{a^i\gamma}{1 + b^i\gamma}n = s^{tr} - \sum\frac{\beta^i_n}{1 + b^i\gamma}, \qquad \left|\eta\right| + \sqrt{6}G\gamma + \sum\frac{a^i\gamma}{1 + b^i\gamma} = \left|s^{tr} - \sum\frac{\beta^i_n}{1 + b^i\gamma}\right|.$$

Eventually,

$$\left|\eta\right| = \left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right| - \sqrt{6}G\gamma - \sum \frac{a^i\gamma}{1+b^i\gamma}. \tag{8}$$

Then η can be expressed as,

$$\eta = \frac{\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right| - \sqrt{6}G\gamma - \sum \frac{a^i\gamma}{1+b^i\gamma}}{\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right|} \left(s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right).$$

It is equivalent to

$$\eta = \left(\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right| - \sqrt{6}G\gamma - \sum \frac{a^i\gamma}{1+b^i\gamma}\right) u.$$

It is easy to see that $n = u$ with $u = \dfrac{s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}}{\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right|}$. The derivatives of u are

$$\frac{\partial u}{\partial \gamma} = \frac{1}{\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right|} \sum \frac{b^i}{(1+b^i\gamma)^2} \left(\beta_n^i - u : \beta_n^i u\right), \qquad \frac{\partial u}{\partial \varepsilon^{tr}} = \frac{1}{\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right|} 2G\left(I_d - u \otimes u\right). \tag{9}$$

5 Scalar Equation Iteration

The yield function is then

$$F = \sqrt{\frac{3}{2}} \left(\left|s^{tr} - \sum \frac{\beta_n^i}{1+b^i\gamma}\right| - \sqrt{6}G\gamma - \sum \frac{a^i\gamma}{1+b^i\gamma}\right) - \left(\sigma_y + k_l\left(p_n + \gamma\right) + k_s\left(1 - e^{-m(p_n+\gamma)}\right)\right). \tag{10}$$

The corresponding derivative is then

$$\frac{\partial F}{\partial \gamma} = \sqrt{\frac{3}{2}} \sum \frac{b^i u : \beta_n^i - a^i}{(1+b^i\gamma)^2} - 3G - k_l - k_s m e^{-m(p_n+\gamma)}, \qquad \frac{\partial F}{\partial \varepsilon^{tr}} = \sqrt{6}Gu : I_d = \sqrt{6}Gu. \tag{11}$$

Note $u : I_d = u$ since u is a deviator.

6 Consistent Tangent Stiffness

For stiffness, ε^{tr} is now varying, then

$$\frac{\partial F}{\partial \varepsilon^{tr}} + \frac{\partial F}{\partial \gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon^{tr}} = 0, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon^{tr}} = -\left(\frac{\partial F}{\partial \gamma}\right)^{-1} \frac{\partial F}{\partial \varepsilon^{tr}}. \tag{12}$$

Since the stress can be written as

$$\sigma = E(\varepsilon^{tr} - \varepsilon^p) = E(\varepsilon^{tr} - \varepsilon^p_n - \Delta\varepsilon^p) = E(\varepsilon^{tr} - \varepsilon^p_n) - \sqrt{6}G\gamma u. \tag{13}$$

The derivative is

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon^{tr}} &= E - \sqrt{6}G\left(\gamma\frac{\partial u}{\partial \varepsilon^{tr}} + \left(u + \gamma\frac{\partial u}{\partial \gamma}\right)\frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon^{tr}}\right) \\ &= E - \sqrt{6}G\gamma\frac{\partial u}{\partial \varepsilon^{tr}} + 6G^2\left(u + \gamma\frac{\partial u}{\partial \gamma}\right)\left(\frac{\partial F}{\partial \gamma}\right)^{-1} u. \end{aligned} \tag{14}$$